Journal of Approximation Theory **111**, 298–314 (2001)
doi:10.1006/jath.2001.3578, available online at http://www.idealibrary.com on IDEAL®

An Asymptotic Expansion of the Double Gamma Function

Chelo Ferreira

Departamento de Matématica Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50013 Zaragoza, Spain
E-mail: cferrei@posta.unizar.es

and

José L. López

Departamento de Matématica e Informática, Universidad Pública de Navarra, 31006 Pamplona, Spain
E-mail: jl.lopez@unavarra.es

Communicated by Alphonse P. Magnus

Received March 23, 2000 ; accepted in revised form February 20, 2001; published online June 20, 2001

The Barnes double gamma function $G(z)$ is considered for large argument z. A new integral representation is obtained for $\log G(z)$. An asymptotic expansion in decreasing powers of z and uniformly valid for $|\operatorname{Arg} z| < \pi$ is derived from this integral. The expansion is accompanied by an error bound at any order of the approximation. Numerical experiments show that this bound is very accurate for real z. The accuracy of the error bound decreases for increasing $\operatorname{Arg} z$. © 2001 Academic Press

Key Words: Barnes double gamma function; asymptotic expansions.

1. INTRODUCTION

The double and generalized double gamma functions were introduced and primarily investigated by Barnes [2–4], who applied these functions to the theory of elliptic and theta functions. Some other mathematical applications of the double gamma function may be found in [9, 11, 20]. The double gamma function is used in [20] to prove the classical Kronecker limit formula. Its p-adic analytic extension appeared in a formula of Cassou–Nogués for the p-adic L functions at the origin [7]. On the other hand, the theory of the double gamma function is used in [9, 11, 21] to evaluate some series involving the zeta function.

0021-9045/01 $35.00

A second field of applications is found in the study of determinants of Laplacians. In fact, multiple gamma functions evaluated at 1/2 may be expressed in terms of the functional determinant of Laplacians of the n-sphere, which have been a recent subject of research due to their relevance to superstring theory [23]. Toeplitz determinants with special rational generating functions may be evaluated in terms of the double gamma function and the Gauss hypergeometric function [5]. Several properties of the gamma and double gamma functions may be deduced from the application of the zeta regularization to the determinants of certain operators [24]. The double gamma function plays a key role in deriving the determinant of the Laplacian on spinor fields on a Riemann surface in terms of the value of a Selberg zeta function at the middle of the critical strip [19]. Some other applications of the double gamma function to the study of determinants of Laplacians may be found in [8, 16, 18].

The double gamma function $G(z)$ is the integral function defined by the infinite product [2]

$$G(z+1)=(2\pi)^{z/2}\, e^{-[(1+\gamma)\, z^2+z]/2} \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right)^k e^{-z+z^2/2k} \right], \tag{1}$$

where γ is the Euler–Mascheroni constant. It satisfies the recursion relation $G(z+1)=\Gamma(z)\, G(z)$ and, for integer positive z, it verifies $G(1)=G(2)=1$ and $G(m)=1!\; 2!...(m-3)!\; (m-2)!$ for $m\geqslant 2$.

Although useful in many applications, the double gamma function $G(z)$ has not appeared in the tables of the most well-known special functions and is just cited in an exercise proposed by Whitaker and Watson [25, p. 264] and used by Gradshteyn and Ryzhik [12, p. 661, Eq. 6.441(2); p. 937, Eq. 8.333].

However, motivated by its recent increasing interest, Richard Askey proposed recently, in the panel discussion of the San Diego symposium on asymptotics and applied analysis (January, 2000), a deeper study of this function. In particular, the derivation of asymptotic expansions of $G(z)$ is necessary for approximating this function at large values of z.

Asymptotic expansions of some functions that generalize the double gamma function have been obtained by Matsumoto [13, 14] and Billingham and King [6]. Matsumoto studies among other things, the function $\Gamma_2(z, (1, w))$ introduced by Barnes in [4] as a generalization of the double gamma function: $G(z)=(2\pi)^{z/2}\, \Gamma_2^{-1}(z, (1, 1))$. Billingham and King study the generalized double gamma function $\bar{G}(z, \tau)$, which satisfies the generalized recursion relation $\bar{G}(z+1, \tau)=\Gamma(z/\tau)\, \bar{G}(z, \tau)$ and the normalization condition $\bar{G}(1, \tau)=1$. It was introduced by Barnes in [3] as a different generalization of the double gamma function: $G(z)=\bar{G}(z, 1)$.

A complete asymptotic expansion of $\log \Gamma_2(z, (1, w))$ is obtained in [13] and [14] from an integral representation of this function. But this is an expansion in decreasing powers of w and therefore, the asymptotic expansion of $G(z)$ in decreasing powers of z cannot be obtained from Matsumoto's expansion.

The first terms of the asymptotic expansions of the generalized double gamma function $\bar{G}(z, \tau)$ for large or small z or τ have been obtained in [6] by using the method of matched asymptotic expansions to solve the difference equation. The first terms of uniform asymptotic expansions are also obtained there. Therefore, an asymptotic approximation of $\log G(z) = \log \bar{G}(z, 1)$ for small or large z may be obtained from the asymptotic approximation of $\log \bar{G}(z, \tau)$ for small or large z, respectively, and fixed $\tau = 1$.

Nevertheless, for small z, several complete convergent expansions of $\log G(z)$ in powers of z are given in [9, Eqs. (2.1), (2.15), (2.25)]. On the other hand, complete asymptotic expansions of $\log G(z)$ for large z are not known. The purpose of this paper is to obtain a complete asymptotic expansion of $\log G(z)$ for large z with error bounds.

In Section 2, we obtain an integral representation of $\log G(z)$ from which we derive a complete asymptotic expansion of $\log G(z)$ for large z. In Section 3 we use the error test and Cauchy's formula for obtaining accurate error bounds at any order of the approximation. Numerical examples are shown as an illustration. A brief summary and a few comments are given in Section 4.

2. ASYMPTOTIC EXPANSION

The starting point for deriving an asymptotic expansion of $\log G(z)$ is a suitable integral representation. It may be obtained from [9, Eq. (2.15)],

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k(k+1)} z^{k+1} = (\log(2\pi) - 1)\frac{z}{2} + (\gamma - 1)\frac{z^2}{2} + z \log \Gamma(z+1) - \log G(z+1), \qquad |z| < 1. \tag{2}$$

Introducing the integral representation of the zeta function [1, Eq. 23.2.7] into the left hand side of the above equation, interchanging sum and integral, and after trivial manipulations we obtain

$$\log G(z+1) = (\log(2\pi) - 1)\frac{z}{2} + (\gamma - 1)\frac{z^2}{2} + z \log \Gamma(z+1) + I(z), \qquad |z| < 1, \tag{3}$$

where $I(z)$ is the integral

$$I(z) \equiv \int_0^\infty \frac{e^{-zx} - 1 + xz - (xz)^2/2}{x^2(e^x - 1)}\, dx.$$

This integral defines an analytic function of z for $\mathrm{Re}(z) > -1$ [22, p. 30, Theorem 2.3]. Therefore, the right hand side of (3) defines the analytic continuation of $\log G(z+1)$ to $\mathrm{Re}(z) > -1$.

The asymptotic expansion of $I(z)$ for large z then provides an asymptotic expansion of $\log G(z+1)$. For obtaining an asymptotic expansion of $I(z)$ we divide the integral into two pieces

$$I(z) = I_1(z) + I_2(z),$$

where

$$I_1(z) \equiv \int_0^\infty \left[\frac{-1 + xz - (xz)^2/2}{x^2(e^x - 1)} + \frac{e^{-zx}}{x^3} - \frac{e^{-zx}}{2x^2} + \frac{e^{-zx}}{12x} \right] dx,$$

$$I_2(z) \equiv \int_0^\infty \frac{e^{-zx}}{x^3} \left[\frac{x}{e^x - 1} - \sum_{k=0}^{2} \frac{B_k}{k!} x^k \right] dx,$$

and B_k are the Bernoulli numbers. The integral $I_1(z)$ may be evaluated by means of elementary techniques by writing

$$I_1(z) = \lim_{\varepsilon \to 0} \left\{ -\int_\varepsilon^\infty \frac{dx}{x^2(e^x - 1)} + z \int_\varepsilon^\infty \frac{dx}{x(e^x - 1)} - \frac{z^2}{2} \int_\varepsilon^\infty \frac{dx}{(e^x - 1)} \right.$$
$$\left. + \int_\varepsilon^\infty \frac{e^{-zx}}{x^3}\, dx - \frac{1}{2} \int_\varepsilon^\infty \frac{e^{-zx}}{x^2}\, dx + \frac{1}{12} \int_\varepsilon^\infty \frac{e^{-zx}}{x}\, dx \right\}. \tag{4}$$

Integrating by parts we have, for $\varepsilon \to 0$,

$$\int_\varepsilon^\infty \frac{e^{-zx}}{x}\, dx = -\log \varepsilon - \log z - \gamma + \mathcal{O}(\varepsilon),$$

$$\int_\varepsilon^\infty \frac{e^{-zx}}{x^2}\, dx = \frac{1}{\varepsilon} + z(\log \varepsilon + \log z + \gamma - 1) + \mathcal{O}(\varepsilon),$$

$$\int_\varepsilon^\infty \frac{e^{-zx}}{x^3}\, dx = \frac{1}{2\varepsilon^2} - \frac{z}{\varepsilon} + \frac{z^2}{2}\left(\frac{3}{2} - \log \varepsilon - \log z - \gamma \right) + \mathcal{O}(\varepsilon).$$

On the other hand, with the change of variable $x = \log y$,

$$\int_\varepsilon^\infty \frac{dx}{e^x - 1} = -\log \varepsilon + \mathcal{O}(\varepsilon).$$

In order to calculate the two remaining integrals in (4) we separate their integrand in two terms

$$\frac{1}{e^x-1}\equiv\frac{1}{x}+\frac{1-e^x+x}{x(e^x-1)}$$

and expand $1-e^x+x$ in power series of x. Using [1, Eq. 23.2.7] we obtain

$$\int_\varepsilon^\infty \frac{dx}{x(e^x-1)}=\frac{1}{\varepsilon}+\frac{1}{2}\log\varepsilon-\sum_{n=2}^{\infty}\frac{\zeta(n)}{n(n+1)}+\mathcal{O}(\varepsilon),$$

$$\int_\varepsilon^\infty \frac{dx}{x^2(e^x-1)}=\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}-\frac{1}{12}\log\varepsilon$$

$$+\frac{1}{2}\sum_{n=2}^{\infty}\frac{\zeta(n)}{n(n+1)}-\sum_{n=2}^{\infty}\frac{\zeta(n)}{n(n+1)(n+2)}+\mathcal{O}(\varepsilon).$$

Taking the limit $z\to -1$ in (2),

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{n(n+1)}=\frac{1}{2}(\log(2\pi)-\gamma).$$

Introducing these calculations in (4) we find

$$I_1(z)=\left(\frac{3}{4}-\frac{\gamma}{2}\right)z^2+\frac{1}{2}(1-\log 2\pi)\,z-\frac{1}{2}\left(z^2+z+\frac{1}{6}\right)\log z+C, \tag{5}$$

where the constant C is defined by

$$C\equiv\frac{\gamma}{6}-\frac{\log(2\pi)}{4}+\sum_{n=2}^{\infty}\frac{\zeta(n)}{n(n+1)(n+2)}. \tag{6}$$

Integrating (2) with respect to z and taking again the limit $z\to -1$ we have

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{n(n+1)(n+2)}$$

$$=\frac{\log(2\pi)}{4}-\frac{\gamma}{6}-\frac{1}{12}-\int_{-1}^{0} t\log\Gamma(t+1)\,dt+\int_{-1}^{0}\log G(t+1)\,dt.$$

Using now [1, Eqs. 6.1.3 and 6.1.41], the definition (1) of $G(z)$ and [17, p. 647, Eq. 1], we obtain $C = -\log A$, where A is Glaisher's constant defined by

$$\log A \equiv \lim_{n\to\infty} \left\{ \log\left(\prod_{k=1}^{n} k^k\right) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right) \log n + \frac{n^2}{4} \right\}. \tag{7}$$

A more stable numerical algorithm useful for evaluating $\log A$ is given by (6). Another one can be obtained by using the Euler–Maclaurin formula [26, p. 36] on the right hand side of (7),

$$\log A = \frac{1}{4} + \frac{1}{12} \int_1^\infty \frac{B_4 - B_4(x - \lfloor x \rfloor)}{x^3}\, dx = \frac{1}{4}$$

$$+ \frac{1}{12} \sum_{n=1}^{\infty} \left[(6n^2 + 6n + 1) \log\left(\frac{n}{n+1}\right) + 6n + 3 \right]$$

$$\simeq 0.2487544770337843.$$

On the one hand, the function $I_1(z)$ given in (5) is analytic in $|\mathrm{Arg}(z)| < \pi$. On the other hand, by using the Cauchy's residua theorem we obtain that the integral $I_2(z)$ may also be written as

$$I_2(z) = \int_0^{\infty e^{i\varphi}} \frac{e^{-zx}}{x^3} \left[\frac{x}{e^x - 1} - \sum_{k=0}^{2} \frac{B_k}{k!} x^k \right] dx, \tag{8}$$

where φ is any angle verifying $|\varphi| < \pi/2$ and $-\pi/2 - \varphi \leqslant \mathrm{Arg}(z) \leqslant \pi/2 - \varphi$. Therefore, the right hand side of (5) plus the right hand side of (8) define the analytic continuation of $\log G(z+1)$ to the sector $|\mathrm{Arg}(z)| < \pi$ if we define $I_2(z)$ by (8) with φ verifying $-\pi/2 - \varphi \leqslant \mathrm{Arg}(z) \leqslant \pi/2 - \varphi$.

Once we have calculated $I_1(z)$ exactly, $I_2(z)$ must be approximated asymptotically. For that purpose we substitute $x/(e^x - 1)$ in this integral by the expansion [1, Eq. 23.1.1],

$$\frac{x}{e^x - 1} = \sum_{n=0}^{N-1} \frac{B_n}{n!} x^n + r_N(x), \qquad N = 1, 2, 3, 5, 7, 9, ..., \tag{9}$$

where $x \neq \pm 2m\pi i$, $m \in \mathbb{N}$, and $r_N(x) = \mathcal{O}(x^N)$ as $x \to 0$, and we obtain

$$I_2(z) = \sum_{n=1}^{N-1} \frac{B_{2n+2}}{2n(2n+1)(2n+2)\, z^{2n}} + R_N(z), \qquad N = 1, 2, 3, ..., \tag{10}$$

where

$$R_N(z)=\int_0^{\infty e^{i\varphi}} \frac{e^{-zx}}{x^3}\, r_{2N+2}(x)\, dx. \tag{11}$$

Therefore, plugging the expansion of $I_2(z)$ and the value of $I_1(z)$ into $I(z)$ and this into Eq. (3), we obtain the following theorem.

THEOREM 1. *For $|\mathrm{Arg}(z)|<\pi$, the logarithm of the double gamma function admits the expansion*

$$\log G(z+1)=\frac{1}{4}z^2+z\log\Gamma(z+1)-\left(\frac{1}{2}z^2+\frac{1}{2}z+\frac{1}{12}\right)\log z-\log A$$
$$+\sum_{n=1}^{N-1}\frac{B_{2n+2}}{2n(2n+1)(2n+2)\,z^{2n}}+R_N(z),\qquad N=1,2,3,\ldots. \tag{12}$$

In the following section we obtain bounds for $R_N(z)$ which show that in fact (12) defines an asymptotic expansion of $\log G(z+1)$.

3. ERROR BOUNDS

We now derive error bounds for $R_N(z)$ at any order N of the approximation (12). We obtain first error bounds for $r_N(x)$ in three different regions of the variable x: $0\leqslant x<2\pi$, $|x|<2\pi$ and the shaded region depicted in Fig. 1. Then, we introduce these bounds into the integral (11) defining $R_N(z)$.

3.1. *Bounds for $r_N(x)$*

For $0\leqslant x<2\pi$, Eq. (9) defines a convergent expansion and therefore, for $N=1,2,3,\ldots$,

$$r_{2N+2}(x)=r_N^{(1)}(x)+r_N^{(2)}(x),$$

where

$$r_N^{(1)}(x)\equiv\sum_{n=0}^{\infty}\frac{B_{4n+2N+2}}{(4n+2N+2)!}\,x^{4n+2N+2}$$

and

$$r_N^{(2)}(x) \equiv \sum_{n=0}^{\infty} \frac{B_{4n+2N+4}}{(4n+2N+4)!}\, x^{4n+2N+4}.$$

For real $x < 2\pi$, all the terms in the sum defining $r_N^{(1)}(x)$ are negative for odd N and positive for even N, whereas the terms in the sum defining $r_N^{(2)}(x)$ are all positive for odd N and negative for even N. Using the bounds for the Bernoulli numbers given in [1, Eq. 23.1.15] we find

$$\begin{aligned}|r_N^{(2)}(x)| &\leqslant 2 \sum_{n=0}^{\infty} \left(\frac{x}{2\pi}\right)^{4n+2N+4} \frac{1}{1-2^{-4n-2N-3}} \\ &\leqslant \frac{2(x/2\pi)^{2N+4}}{1-(x/2\pi)^4} + \frac{2(x/2\pi)^{2N+4}}{(2^{2N+3}-1)(1-(x/4\pi)^4)} \\ &\leqslant \frac{2(x/2\pi)^{2N+2}}{1-(x/2\pi)^4} \leqslant |r_N^{(1)}(x)|.\end{aligned}$$

The third inequality above may be derived by observing that the inequality

$$\left(1+\frac{2^{2N+3}-1}{16}\right) a^4 + a^2 - 2^{2N+3} + 1 \leqslant 0$$

holds for $0 \leqslant a \equiv x/2\pi < 1$ and $N = 1, 2, 3, \ldots$. Therefore we find that $\operatorname{sign}(r_{2N+2}(x)) = (-1)^N$ for $0 \leqslant x < 2\pi$ and $N = 1, 2, 3, \ldots$ and two consecutive error terms in the expansion (9) have opposite signs. Then, applying the error test (see, for example, [15, p. 68] or [26, p. 38]) we find

$$|r_{2N+2}(x)| \leqslant \frac{|B_{2N+2}|}{(2N+2)!}\, x^{2N+2}, \qquad 0 \leqslant x < 2\pi, \qquad N = 1, 2, 3, \ldots. \tag{13}$$

On the other hand, for complex x with $|x| < 2\pi$ we have, using [1, Eq. 23.1.15],

$$\begin{aligned}|r_{2N+2}(x)| &\leqslant \sum_{n=N}^{\infty} \frac{|B_{2n+2}|}{(2n+2)!}\, |x|^{2n+2} \\ &\leqslant 2\left|\frac{x}{2\pi}\right|^{2N+2} \left(\frac{1}{1-|x/2\pi|} + \frac{(2^{2N+1}-1)^{-1}}{1-|x/4\pi|}\right).\end{aligned} \tag{14}$$

Finally, for any $x \in \mathbb{C}$ we consider the explicit expression for $r_{2N+2}(x)$ given by the Lagrange form for the remainder of the Taylor expansion (9),

$$r_{2N+2}(x) = \frac{1}{(2N+2)!} \frac{d^{2N+2}}{dx^{2N+2}} \left(\frac{x}{e^x - 1} \right) \bigg|_{x=\xi} x^{2N+2}, \qquad N = 1, 2, 3, ...,$$

where $\xi \in (0, x)$. By the Cauchy's theorem,

$$r_{2N+2}(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{w \, dw}{(w-\xi)^{2N+3} (e^w - 1)} x^{2N+2}, \qquad N = 1, 2, 3, ...,$$

where $\mathscr{C}$ is a circle with center at the point ξ that does not enclose singularities of $(e^w - 1)^{-1}$. In order to find bounds of $R_N(z)$ we require bounds of $r_{2N+2}(x)$ valid for fixed $\mathrm{Arg}(x) = \varphi$ with $|\varphi| < \pi/2$ and $0 \leqslant |x| < \infty$. Therefore, we make the change of variable $w = \xi + \pi \cos \varphi e^{i\theta}$ (observe that $\pi \cos \varphi <$ the distance of the x-axis to the first singularities $\pm 2\pi i$ of $(e^w - 1)^{-1}$; see Fig. 1), obtaining

$$|r_{2N+2}(x)| \leqslant C(\varphi) \frac{x^{2N+2}}{(\pi \cos \varphi)^{2N+2}}, \qquad N = 1, 2, 3, ..., \tag{15}$$

where $C(\varphi)$ is a bound of $|w/(e^w - 1)|$ in the shaded region depicted in Fig. 1.

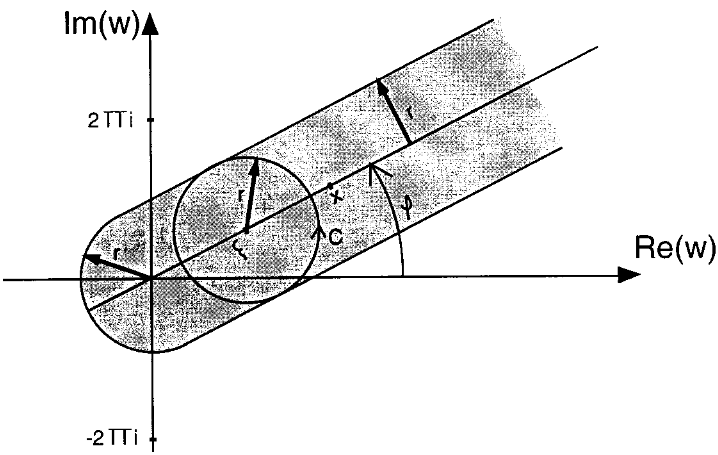

FIG. 1. The circle of radius $r \equiv \pi \cos \varphi$ centered at ξ, with $\xi \in (0, x)$, used in the Cauchy definition of $r_{2N+2}(x)$ must be contained in the shaded region. This region is defined by the set $\{w \in \mathbb{C}, |w - x| < \pi \cos \varphi, 0 \leqslant |x| < \infty\}$.

The maximum of the function $|w/(e^w-1)|$ in the shaded region of Fig. 1 is situated at a tangent point of the level lines of that function with the curve $w(x) \equiv x+iy(x)$ limiting that region,

$$y(x)=\begin{cases} x\tan\varphi+\pi & \text{if} \quad -\pi\sin\varphi\cos\varphi \leqslant x<\infty, \\ \sqrt{\pi^2\cos^2\varphi-x^2} & \text{if} \quad -\pi\cos\varphi\leqslant x< -\pi\sin\varphi\cos\varphi, \\ -\sqrt{\pi^2\cos^2\varphi-x^2} & \text{if} \quad -\pi\cos\varphi\leqslant x<\pi\sin\varphi\cos\varphi, \\ x\tan\varphi-\pi & \text{if} \quad \pi\sin\varphi\ \cos\varphi\leqslant x<\infty. \end{cases}$$

Therefore, we can take the constant $C(\varphi)$,

$$C(\varphi)=\operatorname{Max}_{x_0,\,x_1}\left\{\frac{\sqrt{x_0^2+\pi^2\cos^2\varphi\pm 2x_0\pi\sin\varphi\cos\varphi}}{\sin(x_0\tan\varphi)},\right.$$
$$\left.\frac{\pi\cos\varphi}{\sqrt{e^{2x_1}+1-2e^{x_1}\cos(\sqrt{\pi^2\cos^2(\varphi)-x_1^2})}}\right\}, \tag{16}$$

where x_0 are the solutions of the equation $\tan\varphi\sin(x\tan\varphi)=e^x+\cos(x\tan\varphi)$ in the interval $[-\pi\,|\sin\varphi|\cos\varphi,\infty)$ and x_1 are the solutions of the equations

$$x\sin\sqrt{\pi^2\cos^2\varphi-x^2}=\pm(e^x-\cos\sqrt{\pi^2\cos^2\varphi-x^2})\sqrt{\pi^2\cos^2\varphi-x^2}$$

in the respective intervals $[-\pi\cos\varphi,\mp\pi\sin\varphi\cos\varphi)$.

3.2. *Bounds for $R_N(z)$*

For $|\operatorname{Arg}(z)|<\pi/2$ we can take $\varphi=0$ in (11) and therefore, we can plug the bounds (13) and (15) into the integral (11) defining $R_N(z)$. We can use (13) for $0\leqslant x<2\pi$ and (15) for $x\geqslant 2\pi$ with $C(0)=\pi/(1-e^{-\pi})$, obtaining

$$|R_N(z)|\leqslant\frac{|B_{2N+2}|}{(2N+2)!}\int_0^\infty e^{-x\operatorname{Re}(z)}x^{2N-1}\,dx$$
$$+\int_{2\pi}^\infty e^{-x\operatorname{Re}(z)}x^{2N-1}\left[\frac{1}{(1-e^{-\pi})\,\pi^{2N+1}}-\frac{|B_{2N+2}|}{(2N+2)!}\right]dx,$$
$$N=1,2,3,\ldots.$$

Therefore, we obtaining the following theorem,

THEOREM 2. *For $|\operatorname{Arg}(z)|<\pi/2$, an error bound for the remainder $R_N(z)$ in the expansion* (12) *of the logarithm of the double gamma function is given by*

$$|R_N(z)|\leqslant\frac{C_N(z)}{(\operatorname{Re}(z))^{2N}},\qquad N=1,2,3,\ldots, \tag{17}$$

where $C_N(z)$ *is given by*

$$C_N(z) = \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)} + e^{-2\pi \operatorname{Re}(z)} \left[\frac{1}{(1-e^{-\pi})\, \pi^{2N+1}} - \frac{|B_{2N+2}|}{(2N+2)!} \right] \times \sum_{k=0}^{2N-1} \binom{2N-1}{k} k! (2\pi \operatorname{Re}(z))^{2N-k-1}.$$

The bound (17) is not satisfactory for $\operatorname{Arg}(z)$ close to $\pm\pi/2$ and not valid for $\pi/2 \leqslant |\operatorname{Arg}(z)| < \pi$. A more accurate error bound may be obtained for $\operatorname{Arg}(z)$ close to $\pm\pi/2$ which is valid also in the whole sector $|\operatorname{Arg}(z)| < \pi$ by using the freedom we have when choosing the parameter φ in (11). For that purpose we introduce in (11) the bound (14) for $|x| \leqslant \pi$ and the bound (15) for $|x| > \pi$. After trivial manipulations and choosing $\varphi = -\frac{1}{2} \operatorname{Arg}(z)$, we find the following theorem:

THEOREM 3. *For* $|\operatorname{Arg}(z)| < \pi$, *an error bound for the remainder* $R_N(z)$ *in the expansion* (12) *of the logarithm of the double gamma function is given by*

$$|R_N(z)| \leqslant \frac{C_N(z)}{|z \cos(\operatorname{Arg}(z)/2)|^{2N}}, \qquad N = 1, 2, 3, ..., \tag{18}$$

where

$$C_N(z) \equiv \frac{4(2N-1)!}{(2\pi)^{2N+2}} \left(1 + \frac{2}{3(2^{2N+1}-1)}\right) + \left[\frac{C(-\operatorname{Arg}(z)/2)}{(\pi \cos(\operatorname{Arg}(z)/2))^{2N+2}} - \frac{4}{(2\pi)^{2N+2}} \left(1 + \frac{2}{3(2^{2N+1}-1)}\right)\right] \times e^{-\pi\, |z \cos(\operatorname{Arg}(z)/2)|} \sum_{k=0}^{2N-1} \binom{2N-1}{k} k! \left(\pi\, |z| \cos\left(\frac{\operatorname{Arg}(z)}{2}\right)\right)^{2N-1-k}$$

and $C(\varphi)$ *is given in* (16).

This bound shows that, in fact, expansion (12) is an asymptotic expansion of $\log G(z+1)$ in the sector $|\operatorname{Arg}(z)| < \pi$.

Tables I–IV show numerical experiments about the approximation supplied by expansion (12) for some values of z and the accuracy of the error bounds (17) and (18).

In Tables I and II the second, third and sixth columns represent the integral $I_2(z) = \log G(z+1) - \frac{1}{4}z^2 - z \log \Gamma(z+1) + (\frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{12}) \log z + \log A$,

TABLE I

$(\mathrm{Arg}(z) = 0)$

$\lvert z \rvert$	$I_2(z)$	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
1	−0.0012455229	−0.0013888888	0.115	2.24	−0.0011904761	0.0442	13.4
2	−0.0003360773	−0.0003472222	0.0332	0.0425	−0.0003348214	0.00374	0.032
5	−0.0000552441	−0.0000555555	0.00564	0.00574	−0.0000552380	0.000110	0.000115
10	−0.0000138691	−0.0000138888	0.00142	0.00143	−0.0000138690	7.009e-6	7.18e-6
20	−3.4709876e-6	−3.4722222e-6	0.0003568	0.0003572	−3.4709821e-6	4.45e-7	4.47e-7
50	−5.55552382e-7	−5.5555555e-7	5.713e-5	5.715e-5	−5.55552381e-7	1.142e-8	1.143e-8
100	−1.38886905e-7	−1.38888888e-7	1.42852e-5	1.42859e-5	−1.38886905e-7	7.142e-10	7.143e-10

TABLE II

$(\mathrm{Arg}(z) = \pi/4)$

$\|z\|$	$I_2(z)$	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
5	−3.171562e-7 + .000055554i	.000055555i	.00571	..0159	−3.174603e-7 + .000055555i	.000125	..000539
10	−1.976706e-8 + .000013886i	.000013888i	.00144	..00394	−1.984126e-8 + .000013888i	7.14e-6	..000023
20	−1.240075e-9 + 3.47777e-6i	3.472222e-6i	.000357	..000984	−1.240079e-9 + 3.47777e-6i	4.46e-7	1.43e-6
50	−3.1746029e-11 + 5.555555492e-7i	5.555555555e-7i	.0000571	..000157	−3.1746031e-11 + 5.555555555e-7i	1.14e-8	3.67e-8
100	−1.984126973e-12 + 1.3888888887e-7i	1.38888888e-7i	1.14e-8	..0000394	−1.984126984e-12 + 1.38888888e-7i	7.14e-10	2.3e-9

TABLE III

$(\mathrm{Arg}(z) = \pi/2)$

$\|z\|$	$I_2(z)$	First order approx.	Relative error	Second order approx.	Relative error
5	.000056023	.000055555	.0111	..000055873	.000119
10	.000013908	.000013888	..00143	.000013908	7.21e-6
20	3.473463e-6	3.472222e-6	.000357	3.473462e-6	4.47e-7
50	5.555873079e-7	5.55555555e-7	.0000572	5.555873016e-7	1.14e-8
100	1.3889087311e-7	1.388888888e-7	.0000143	1.3889087301e-7	7.14e-10

approximation (10) for $N=2$ and approximation (10) for $N=3$ respectively. Fourth and seventh columns represent the respective relative error $-R_N(z)/I_2(z)$. Fifth and last columns represent the respective relative error bounds given by Eqs. (17) or (18).

In Tables III and IV the second, third and fifth columns represent the integral $I_2(z) = \log G(z+1) - \frac{1}{4}z^2 - z\log\Gamma(z+1) + (\frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{12})\log z + \log A$, approximation (10) for $N=2$ and approximation (10) for $N=3$ respectively. Fourth and sixth columns represent the respective relative error $-R_N(z)/I_2(z)$.

4. CONCLUSIONS

The coefficients of the analytic expansion at $z=0$ of the function in the right hand side of (2) (which involve the logarithm of the double gamma function $G(z)$) are given in terms of the zeta function. After introducing an integral representation of the zeta function in this formula, we have derived the integral representation (8) for the function given in the right hand side of (2). By analytical continuation, this integral representation is valid for $|\mathrm{Arg}(z)| < \pi$. Introducing the expansion of the integrand (9) into this integral, we have obtained the asymptotic expansion (12) of $\log G(z+1)$ uniformly valid for $|\mathrm{Arg}(z)| \leqslant \pi - \delta < \pi$. The expansion (9) verifies the error test for real $0 \leqslant x < 2\pi$ and then an accurate error bound (17) has been obtained for the remainder in the expansion (12) for real z. The bound (17) is not satisfactory for $\mathrm{Arg}(z)$ close to $\pm\pi/2$ and not valid for $\pi/2 \leqslant |\mathrm{Arg}(z)| < \pi$. Using the freedom we have when choosing the parameter φ in (11), a more accurate error bound has been obtained for $\mathrm{Arg}(z)$ close to $\pm\pi/2$. Moreover, this bound is valid for the whole sector $|\mathrm{Arg}(z)| < \pi$. Numerical experiments in Table II show the accuracy of the expansion (12) for real or complex z and the bounds (17) and (18) for real z. Error bounds for complex z are less realistic.

TABLE IV

$(\mathrm{Arg}(z) = 3\pi/4)$

$\|z\|$	$I_2(z)$	First order approx.	Relative error	Second order approx.	Relative error
5	−3.171930e-7 − .0000055549 i	−.000055555 i	.00571	−3.174603e-7 − .000055555 i	.000114
10	−1.984021e-8 − .00001388879 i	−0.00001388889 i	.00143	−1.984126e-8 − .00001388889 i	7.14e-6
20	−1.240075e-9 − 3.472220e-6 i	−3.472222e-6 i	.000357	−1.240079e-9 − 3.472222e-6 i	4.46e-7
50	−3.1746029e-11 − 5.5555554e-7 i	−5.5555555e-7 i	.0000571	−3.17460317e-11 − 5.5555555e-7 i	1.14e-8
100	−1.984126973e-12 − 1.388888888e-7 i	−1.388888889e-7 i	.0000143	−1.984126984e-12 − 1.388888889e-7 i	7.14e-10

ACKNOWLEDGMENTS

This work was stimulated by conversations with Richard Askey and Javier Sesma. The continued interest of Javier Sesma has been quite encouraging. We are grateful to the anonymous referees for their improving suggestions. The financial support of *Comisión Interministerial de Ciencia y Tecnologia* is acknowledged.

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